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A CHARACTERIZATION OF RIEMANN INVARIANTS FOR 2×2 SYSTEM OF HYPERBOLIC CONSERVATION LAWS

V. BOJKOVIC and D. MITROVIC

Faculty of Mathematics
University of Montenegro
Cetinjski Put Bb 81000 Podgorica
Montenegro
e-mail: vlavelbo@cg.yu
matematika@cg.yu

Abstract

Denote by u and v solutions to a strictly hyperbolic 2×2 system of conservation laws. We show that if we assume a functional dependence $v = \alpha(u)$, then the function α "defines" Riemann invariants through an ordinary differential equation for the unknown function α.

1. Introduction

We consider the following system of partial differential equations:

$$\partial_t u + \partial_x f(u, v) = 0,$$

$$\partial_t v + \partial_x g(u, v) = 0, \quad (t, x) \in \mathbf{R}^+ \times \mathbf{R}. \tag{1}$$

The latter system is known as system of conservation laws. If we are

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dealing only with classical solution to the system and rewrite it in the vector form $\partial_t U + \partial_x F(U) = 0$, $U = (u, v)$, $F = (f, g)$, usually we call it a continuity equation. It is differential equation that describes the conservative transport of some kind of quantity. Since mass, energy, momentum, and other natural quantities are conserved, a vast variety of physics may be described with continuity equations. Actually, continuity equations are the (stronger) local form of conservation laws.

It is well known that system (1) can be diagonalized assuming that the solutions u and v are smooth. More precisely, system (1) is equivalent to the following system of continuity equations:

$$\partial_t \omega + \lambda_1(u, v)\partial_x \omega = 0,$$

$$\partial_t \eta + \lambda_2(u, v)\partial_x \eta = 0,$$

where $\omega = \omega(u, v)$ and $\eta = \eta(u, v)$ are so called *Riemann invariants*, and λ_i, $i = 1, 2$, are eigenvalues of the matrix $DF(u, v) = \nabla(f(u, v), \ g(u, v))$. We provide necessary definitions (see e.g. [2, 4] :

Definition 1. By $U = (u, v) \in \Omega \subset \mathbf{R}^2$ we denote classical solutions to system (1). By $\Omega \subset \mathbf{R}^2$ we denote set of possible states of the solution U. By

$$DF(U) = \nabla(f(u, v), \ g(u, v)) = \begin{pmatrix} \partial_u f(u, v) & \partial_u f(u, v) \\ \partial_u g(u, v) & \partial_u g(u, v) \end{pmatrix}$$

we denote Jacobian of the flux vector $F(u, v) = (f(u, v), \ g(u, v))$, $(u, v) \in \Omega$.

By $R_i = R_i(u, v)$, $i = 1, 2$, we denote right eigenvectors of the matrix $DF(U)$.

Definition 2. Riemann invariants $(\omega, \eta) : \Omega \to \mathbf{R}^2$, $\Omega \subset \mathbf{R}^2$, of system (1) are smooth scalar valued functions ω and η such that

$$Dw(U)R_1(U) = 0, \quad D\eta(U)R_2(U) = 0, \qquad U \in \Omega. \tag{2}$$

Definition 3. System of conservation laws (1) is called *hyperbolic* if the matrix $DF(U)$ has two real eigenvalues $\lambda_1(U)$ and $\lambda_2(U)$, and two linearly independent eigenvectors $R_1(U)$ and $R_2(U)$ for every $U \in \Omega$.

Remark 4. Note that, in particular, the linear independency of the vectors $R_1(U)$ and $R_2(U)$ means that it must be $R_i \neq 0$, $i = 1, 2$, for every $U \in \Omega$.

The Riemann invariants are very important tool in investigating many physical problems (see e. g. randomly chosen [3, 4, 5]). In the current paper, by putting $v = \alpha(u)$, we give the characterization of this important notion through an ordinary differential equation. At the end of the paper, we propose similar procedure for $n \times n$ system of conservation laws.

2. Main Result

We want to describe Riemann invariants through an ordinary differential equation. As we shall see, this is equivalent to reducing system (1) to a scalar conservation law. In order to accomplish this we put

$$v = \alpha(u),$$

for an $\alpha \in C^1(\mathbf{R})$. Substituting the latter relation in (1) we get:

$$\begin{aligned} &\partial_t u + \partial_x f(u, \alpha(u)) = 0, \\ &\alpha'(u)\partial_t u + \partial_x g(u, \alpha(u)) = 0, \quad (t, x) \in \mathbf{R}^+ \times \mathbf{R}. \end{aligned} \tag{3}$$

Multiplying the first equation of the system by $\alpha'(u)$ and then subtracting it from the second one we get:

$$(f_u \alpha'(u) + f_u(\alpha'(u))^2)\partial_x u = (g_u + g_v \alpha'(u))\partial_x u, \tag{4}$$

where

$$f_u = \partial_u f(u, v)\big|_{v=\alpha(u)}, \quad f_v = \partial_v f(u, v)\big|_{v=\alpha(u)},$$

$$g_u = \partial_u g(u, v)\big|_{v=\alpha(u)}, \quad g_v = \partial_v g(u, v)\big|_{v=\alpha(u).}$$

From (4) it is natural to take

$$f_u \alpha'(u) + f_v(\alpha'(u))^2 = g_u + g_v \alpha'(u), \tag{5}$$

which is actually the ordinary differential equation mentioned in the abstract. Multiplying (5) by f_v we get:

$$f_v f_u \alpha'(u) + (f_v \alpha'(u))^2 = f_v g_u + f_v g_u \alpha'(u). \tag{6}$$

Putting $Z = f_v \alpha'(u)$ equation (6) becomes:

$$Z^2 + (f_u - g_v)Z - f_v g_u = 0.$$

Solving this simple algebraic equation we get:

$$Z = \frac{-f_u + g_v \pm \sqrt{(f_u - g_v)^2 + 4 f_v g_u}}{2}$$

or, after recalling the form of Z:

$$f_v \alpha'(u) = \frac{-f_u + g_v \pm \sqrt{(f_u - g_v)^2 + 4 f_v g_u}}{2}. \tag{7}$$

Then put $\alpha'(u) = \dfrac{d\alpha}{du}$. As we shall see in (12), from the hyperbolicity of (1), it must be $f_v \neq 0$. Therefore, we get from (7):

$$\frac{d\alpha}{du} = \frac{-f_u + g_v \pm \sqrt{(f_u - g_v)^2 + 4 f_v g_u}}{2 f_v}. \tag{8}$$

For functions $a, b : \mathbf{R}^2 \to \mathbf{R}$, and constants c_1 and c_2, denote by

$$a(u, \alpha) = c_1 \text{ and } b(u, \alpha) = c_2 \tag{9}$$

equipotential manifolds in (u, α)-plane (locally) defining integral curves corresponding to equations (8) with + and -, respectively.

Now, it is easy to prove the following theorem.

Theorem 5. *Riemann invariants* (ω, η) *for system* (1) *are given by*

$$\omega(u, v) = a(u, v), \text{ and } \eta(u, v) = b(u, v), \tag{10}$$

for a and b given by (9)

Proof. It is not difficult to compute right eigenvectors R_i, $i = 1, 2$, to $DF(U)$. It holds:

$$R_1(u, v) = (2f_{v,}\ - f_u + g_v + \sqrt{(f_u - g_v)^2 + 4f_v g_u}),$$

$$R_2(u, v) = (2f_{v,}\ - f_u + g_v - \sqrt{(f_u - g_v)^2 + 4f_v g_u}). \tag{11}$$

Since we assumed that system (1) is strictly hyperbolic it must hold

$$f_v = f_v(u, v) \neq 0, \tag{12}$$

for $(u, v) \in \Omega$, where $\Omega \subset \mathbf{R}^2$ is the set of possible states of the solution (u, v). Indeed, if $f_v(u_0, v_0) = 0$ for an $(u_0, v_0) \in \Omega$, it follows from (11) that either $R_1(u_0, v_0) = 0$ or $R_2(u_0, v_0) = 0$ contradicting hyperbolicity of the system (see Remark 4).

Thus, we can rewrite equations (2) for Riemann invariants in the following form:

$$2f_v \frac{\partial \omega}{\partial u} + \left(- f_u + g_v + \sqrt{(f_u - g_v)^2 + 4f_v g_u}\right)\frac{\partial \omega}{\partial v} = 0,$$

$$2f_v \frac{\partial \eta}{\partial u} + \left(- f_u + g_v - \sqrt{(f_u - g_v)^2 + 4f_v g_u}\right)\frac{\partial \eta}{\partial v} = 0.$$

System of characteristics for the latter linear partial differential equations for ω and η are given by, respectively:

$$\dot{u} = 2f_v, \quad \dot{v} = -f_u + g_v + \sqrt{(f_u - g_v)^2 + 4f_v g_u},$$

$$\dot{u} = 2f_v, \quad \dot{v} = -f_u + g_v - \sqrt{(f_u - g_v)^2 + 4f_v g_u}. \tag{13}$$

Now, we return to ordinary differential equations (8). Introducing a parameter $s \in \mathbf{R}$ and putting $\alpha = \alpha(s)$ and $u = u(s)$ we can rewrite (8) as the following systems:

$$\dot{u} = 2f_v, \quad \dot{\alpha} = -f_u + g_v + \sqrt{(f_u - g_v)^2 + 4f_v g_u};$$

$$\dot{u} = 2f_v, \quad \dot{\alpha} = -f_u + g_v - \sqrt{(f_u - g_v)^2 + 4f_v g_u}; \tag{14}$$

where $\alpha = \alpha(s)$ and $u = u(s)$.

Comparing (14) and (8) we immediately obtain the statement of the theorem.

In what follows, we give a note on a general system of $n \times n$ conservation laws:

$$\partial_t u_1 + \partial_x f_1(u_1, u_2, \ldots, u_n) = 0,$$

$$\partial_t u_2 + \partial_x f_2(u_1, u_2, \ldots, u_n) = 0,$$

$$\vdots$$

$$\partial_t u_n + \partial_x f_n(u_1, u_2, \ldots, u_n) = 0. \tag{15}$$

Similarly as before, we put

$$u_i = g_i(u_1), \quad i = 2, \ldots n,$$

where it is presumed that $g_i \in C^1(\mathbf{R})$. So, (15) is equivalent to

$$\partial_t u_1 + \partial_x f_1(u_1, g_2(u_1), \ldots, g_n(u_1)) = 0,$$

$$g_2'(u_1)\partial_t u_1 + \partial_x f_2(u_1, g_2(u_1), \ldots, g_n(u_1)) = 0,$$

$$\vdots$$

$$g_n'(u_1)\partial_t u_1 + \partial_x f_n(u_1, g_2(u_1), \ldots, g_n(u_1)) = 0.$$

Multiplying the first equation by $g_i'(u_1)$, subtracting ith equation from it, $i = 2, \ldots, n,$ and arguing as in (4), we obtain system:

$$\left(\frac{\partial f_1}{\partial u_1}(u_1, g_2(u_1), \dots, g_n(u_1)) + \cdots + \frac{\partial f_1}{\partial u_n}(u_1, g_2(u_1), \dots, g_n(u_1))g_n'(u_1)\right)g_2'(u_1)$$

$$= \frac{\partial f_2}{\partial u_1}(u_1, g_2(u_1), \dots, g_n(u_1)) + \cdots + \frac{\partial f_2}{\partial u_1}(u_1, g_2(u_1), \dots, g_n(u_1))g_n'(u_1),$$

$$\left(\frac{\partial f_1}{\partial u_1}(u_1, g_2(u_1), \dots, g_n(u_1)) + \cdots + \frac{\partial f_1}{\partial u_n}(u_1, g_2(u_1), \dots, g_n(u_1))g_n'(u_1)\right)g_3'(u_1)$$

$$= \frac{\partial f_3}{\partial u_1}(u_1, g_2(u_1), \dots, g_n(u_1)) + \cdots + \frac{\partial f_3}{\partial u_n}(u_1, g_2(u_1), \dots, g_n(u_1))g_n'(u_1),$$

$$\vdots$$

$$\left(\frac{\partial f_1}{\partial u_1}(u_1, g_2(u_1), \dots, g_n(u_1)) + \cdots + \frac{\partial f_1}{\partial u_n}(u_1, g_2(u_1), \dots, g_n(u_1))g_n'(u_n)\right)g_n'(u_1)$$

$$= \frac{\partial f_n}{\partial u_1}(u_1, g_2(u_1), \dots, g_n(u_1)) + \cdots + \frac{\partial f_n}{\partial u_n}(u_1, g_2(u_1), \dots, g_n(u_1))g_n'(u_1).$$

$$(16)$$

For practical reasons, we introduce notation $\frac{\partial f_i}{\partial u_j}(u_1, g_2(u_1), \dots, g_n(u_1)) = a_{ij}$ and $g_i'(u_1) = x_i$. Now, in terms of linear algebra, system (16) can be rewritten as:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{11}x_2 & a_{12}x_2 & \dots & a_{1n}x_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{11}x_n & a_{12}x_n & \dots & a_{1n}x_n \end{pmatrix} \begin{pmatrix} 1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad (17)$$

It is clear that the latter system does not have to posses any solution. Furthermore it is very complicated (actually impossible in most cases) to find analytic expression for a solution to system (17) if $n > 2$. Therefore, interesting question of finding connection between number of zeros to (17) and existence of the coordinate system of Riemann invariants for (15) remains open if $n > 2$.

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